1. Let \((X, d)\) be a metric space, and let \(\{f_n : X \to \mathbb{R}\}\) be an equicontinuous sequence of functions. Consider the set

\[ C = \{x \in X \mid f_n(x) \text{ converges}\}. \]

a) Prove \(C\) is closed.

b) Let \(X = [-1, 1]\) with the Euclidean metric. Give an example where \(C = \{0\}\).

2. Let \(X\) be a separable Banach space. Show that there is an isometric embedding from \(X\) to \((\ell^\infty, \|\cdot\|_\infty)\) (i.e. the space of bounded sequences).

3. Let \((X, \Sigma, \mu)\) be a measure space, and \(a_k : X \to [0, \infty)\) be a \(\mu\)-measurable function for each \(k\) with

\[ \int_X a_k(x) d\mu \leq \frac{1}{2^k}. \]

Show that the series \(\sum_{k=1}^\infty a_k(x)\) is convergent for \(\mu\)-almost every \(x \in X\).

4. Consider the Hilbert space \(L^2(\mathbb{T})\) of periodic \(L^2\) functions on \([-\pi, \pi]\). For each of the following three sequences, determine and prove whether they converge strongly, weakly but not strongly, or diverge.

\[ f_n(x) = n \cdot \chi_{[0, \frac{1}{n})} \quad g_n(x) = n^{\frac{1}{2}} \cdot \chi_{[0, \frac{1}{n})} \quad h_n(x) = e^{inx}. \]

5. Suppose \(f\) is a bounded monotonic function on \([-\pi, \pi]\). Show that \(f \in H^s(\mathbb{T})\) for any \(0 \leq s < \frac{1}{2}\).

6. Suppose \(f \in C_0^\infty(\mathbb{R})\) such that \(|\frac{d^n f}{dx^n}(x)| \leq 1\) for all \(x \in \mathbb{R}\) for all integers \(n \geq 1\) and \(f(x) = 0\) for all \(|x| > 1\). Show that \(\hat{f}\) (the Fourier transform \(\hat{f}(\xi) = \int_\mathbb{R} f(x) e^{-2\pi i \xi x} dx\)) satisfies the following estimate

\[ |\hat{f}(y)| \leq \frac{2}{(2\pi)^n} |y|^{-n} \text{ for all } y \in \mathbb{R} \text{ and all } n \geq 1. \]